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The Low Energy Effective Equations of Motion for Multibrane Worlds Gravity

Freddy P. Zen,[a] **Arianto,**[a,b] **Bobby E. Gunara,**[a] **Hishamuddin Zainuddin**[c]

[a] *Theoretical Physics Laboratory, Department of Physics,*
Institut Teknologi Bandung
Jl. Ganesha 10 Bandung 40132, Indonesia.

[b] *Department of Physics, Universitas Udayana*
Jl. Kampus Bukit Jimbaran Denpasar 80361, Indonesia.

[c] *Theoretical Studies Laboratory, ITMA, Universiti Putra Malaysia,*
43400 UPM Serdang, Selangor, Malaysia.

E-mail: *fpzen@fi.itb.ac.id, arianto@upk.fi.itb.ac.id, bobby@fi.itb.ac.id, hisham@fsas.upm.edu.my.*

Abstract: The three 3-brane system with both positive or negative tension is studied in a low energy regime by using gradient expansion method. The effective equations of motion on the brane is derived and in particular we examine, in the first order, the radion effective lagrangian for this system. In this case, we show the solution of the modified Friedmann equation with dark radiation on the middle brane and the other 3-branes by direct elimination of the radion fields and Weyl scaling of the metric on the branes. We also derived the scalar-tensor gravity on the branes.

Keywords: Brane World Gravity, Dark Radiation, Scalar-tensor Theory.

Contents

1. Introduction

One of the long standing problem in particle physics and gravitational theories is how to understand quantum theory of gravity. Nowadays, the only possible candidate for this theory is the superstring theory [1]. Interestingly, this theory predicts the existence of the extra dimensions. In order to reconcile this prediction with our observed four dimensional universe, we need a mechanism to compactify the extra dimensions. In particular the setup of heterotic M-theory and its compactification down to five dimensions [2] leads to a well motivated five dimensional brane world scenario, which can be used to study its consequences in particle physics and cosmology.

Randall and Sundrum (RS) proposed two similar but distinct phenomenological brane world scenarios [3, 4]. The first scenario is composed of two branes of the opposite tension, namely RS I. This scenario is the five dimensional space-time which all matter fields are assumed to be confined on branes at fixed points of the S^1/Z_2 orbifold so that the bulk is described by pure Einstein gravity with a negative cosmological constant. On the other hand, the second scenario has a single brane with a positive tension, namely RS II. The fifth dimension is infinite but still Z_2 symmetry

is imposed. In both scenarios the existence of the branes and the bulk cosmological constant makes the bulk geometry curved, or warped.

Furthermore, the brane world models are expected to shed some light onto not only quantum gravity and unification issues, but also cosmological issues such as the cosmological constant or dark energy problem, or even particle physics ones such as the hierarchy problem. There is also a brane world alternative to the standard big bang plus inflation scenario [5, 6]. This scenario consist of a five dimensional bulk bounded by two branes, which are as usual located at the fixed point of the S^1/Z_2 orbifold. Although many brane world models might be over simplified, they should help in learning and understanding the properties of an effective theory derived from some action in space-time dimensions $d > 4$ [7].

There are two approaches to obtain the effective Einstein equations on the brane in the context of Randall-Sundrum scenarios, namely covariant curvature formulation and gradient expansion method. In the covariant formulation [8], the effective Einstein equation can be obtained by projecting the covariant five dimensional Einstein equations on to the brane. The resulting projected equations are modified with respect to general relativity due to the presence of a local quadratic term in the sources and to the presence of a non-local term which is the projection of the five dimensional Weyl tensors. This last term carries information of the bulk gravitational field on the brane and its contribution is of fundamental importance as it might be relevant even at low energy [9, 10, 11, 12].

The main difficulty in understanding the contribution of the projected Weyl tensor to the effective theory on the brane is in its non-local character. The equation for the projected Weyl tensor on the brane are not closed so that solving the full five dimensional equation of motion is necessary.

On the other hand, gradient expansion method gives a way out of this problem. This method first proposed by Kanno and Soda [13]. The main idea is to treat the issue perturbatively, defining a low energy regime in which the energy density on the brane is kept small with respect to bulk vacuum energy density. The perturbation parameter is defined as the ratio between these two energy densities. The five dimensional equations of motion can be solved at different orders in the perturbation parameter. This method allows in principle to derive the effective equations of motion on the brane at each order.

In this paper, we generalize the results obtained for a two 3-brane scenario (RS I) to a multibrane scenario [14, 15, 16] with the set up of the following. We consider the A, B, C 3-branes system, with A and B branes are placed at the fixed points of the orbifold whereas C brane is put between the two fixed points. We derive the effective equations of motion in this scenario. By performing a perturbative expansion of the metric, an expansion of the extrinsic curvature tensor and Weyl tensor are considered. The four basic equations for the five dimensional evolution equations and junction conditions are then solved at different order in the expansion parameter.

The parameter of expansion is determined as in the Anti-deSitter (AdS) scenario. There is a constant scale, namely the AdS curvature scale, to which quantities can be compared.

The paper is organized as follows. In section 2, we explain the general set up for the multibrane system. The junction conditions for this system is derived for each brane by using geometrical approach. In section 3, we give the basic formulation of the gradient expansion method by solving five dimensional equations of motion and imposing the Dirichlet boundary condition at the brane position. We give the basic formulations of the extrinsic curvature at the zeroth order and the first order expansions. The first order solution of the effective equations of motion on the brane is derived in section 4. In section 5, we address our result into the scalar-tensor theory. We conclude our results in section 6.

2. The general setup and background solution

In this section we discuss three 3-brane embedded in a five-dimensional space-time. This model is a straightforward extension of the RS I model. In more detail, we have three parallel 3-branes in an AdS_5 space with negative cosmological constant. The fifth dimension has the geometry of an orbifold S^1/Z_2 and the branes are located at $y = 0$ (A-brane with tension σ_A), $y = y_A$ (C-brane with tension σ_C) and $y = y_B$ (B-brane with tension σ_B). Note that A-brane and B-brane are placed at the fixed points of the orbifold S^1/Z_2 but C-brane is not. The region between two brane, namely region A $[0, y_A]$ and region B $[y_A, y_B]$, is characterized by two different values in two different slice of AdS_5 curvature scales i.e., l_A and l_B, respectively. Furthermore the metrics on the three branes are all connected by an conformal transformation, so in principle it is enough to derive the four-dimensional effective equations of motion on one of the branes. In the following we derive the four-dimensional Einstein equations of motion on each branes. The action that describes this configuration is

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(\mathcal{R} + \frac{12}{l^2_{A,B}}\right) + \sum_{i=A,B,C}\int d^4x\sqrt{-g^{i-brane}}(-\sigma_i + L^i_{matter}), \quad (2.1)$$

where $\mathcal{R}$, κ and g are the scalar curvature, the gravitational constant in 5-dimensions, respectively. The metric $g^{i-brane}$ is the induced metric on the branes and σ_i their tensions. Notice that we have allowed the curvature scales to have different values to either side of the middle brane, represented by l_A and l_B, which leads to different metrics in the two bulk regions. The coordinate system chosen is

$$ds^2 = e^{2\phi(y,x)}dy^2 + g_{\mu\nu}(y,x)dx^\mu dx^\nu. \quad (2.2)$$

The proper distance between two branes with fixed x coordinates can be written as

$$d_A(x) = \int_0^{y_A} dy' e^{\phi(y',x)} \ , \tag{2.3}$$

$$d_B(x) = \int_{y_A}^{y_B} dy' e^{\phi(y',x)} \ . \tag{2.4}$$

And the total proper distance is given by

$$d_{tot}(x) = d_A(x) + d_B(x) = \int_0^{y_B} dy' e^{\phi(y',x)} \ . \tag{2.5}$$

The Einstein equations that arise from this coordinate system are:

$$\begin{aligned} & e^{-\phi}(e^{-\phi}K_\mu{}^\nu)_{,y} - (e^{-\phi}K)(e^{-\phi}K_\mu{}^\nu) + {}^{(4)}R^\nu_\mu - D_\mu D^\nu\phi - D_\mu\phi D^\nu\phi \\ &\quad = -\frac{4}{l^2_{A,B}}\delta^\nu_\mu + \kappa^2\left(\frac{1}{3}\delta^\nu_\mu\sigma_A + T^{A\nu}_\mu - \frac{1}{3}\delta^\nu_\mu T^A\right)e^{-\phi}\delta(y) \\ &\qquad + \frac{\kappa^2}{2}\left(\frac{1}{3}\delta^\nu_\mu\sigma_C + \tilde{T}^{C\nu}_\mu - \frac{1}{3}\delta^\nu_\mu\tilde{T}^C\right)e^{-\phi}\delta(y-y_A) \\ &\qquad + \kappa^2\left(\frac{1}{3}\delta^\nu_\mu\sigma_B + \hat{T}^{B\nu}_\mu - \frac{1}{3}\delta^\nu_\mu\hat{T}^B\right)e^{-\phi}\delta(y-y_B) \ , \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} & e^{-\phi}(e^{-\phi}K)_{,y} - (e^{-\phi}K^{\alpha\beta})(e^{-\phi}K_{\alpha\beta}) - D^\alpha D_\alpha\phi - D^\alpha\phi D_\alpha\phi \\ &\quad = -\frac{4}{l^2_{A,B}} - \frac{\kappa^2}{3}\left(-4\sigma_A + T^A\right)e^{-\phi}\delta(y) - \frac{\kappa^2}{6}\left(-4\sigma_C + \tilde{T}^C\right)e^{-\phi}\delta(y-y_A) \\ &\qquad - \frac{\kappa^2}{3}\left(-4\sigma_B + \hat{T}^B\right)e^{-\phi}\delta(y-y_B) \ , \end{aligned} \tag{2.7}$$

and

$$D_\nu(e^{-\phi}K^\nu_\mu) - D_\mu(e^{-\phi}K) = 0 \ , \tag{2.8}$$

where the appropriate AdS_5 curvature scale is chosen for each region. ${}^{(4)}R^\nu_\mu$ is the curvature on the brane and D_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. When the metric changes between regions of the bulk, it is necessary to include a term in the boundary action that depends on the trace of the extrinsic curvature.

The extrinsic curvature tensor is defined by $K_{\mu\nu} = -g_{\mu\nu,y}/2$. For all three branes, the junction conditions read

$$e^{-\phi}\left[K^\nu_\mu - \delta^\nu_\mu K\right]|_{y=0} = \frac{\kappa^2}{2}\left(-\sigma_A\delta^\nu_\mu + T^{A\nu}_\mu\right) \ , \tag{2.9}$$

$$e^{-\phi}\left[K^\nu_\mu - \delta^\nu_\mu K\right]^+_-|_{y=y_A} = \kappa^2\left(-\sigma_C\delta^\nu_\mu + \tilde{T}^{C\nu}_\mu\right) \ , \tag{2.10}$$

$$e^{-\phi}\left[K^\nu_\mu - \delta^\nu_\mu K\right]|_{y=y_B} = -\frac{\kappa^2}{2}\left(-\sigma_B\delta^\nu_\mu + \hat{T}^{B\nu}_\mu\right) \ , \tag{2.11}$$

where we have used that A-brane and B-brane follow a Z_2 symmetry but C-brane is no Z_2 symmetry. Here, we assume that the direction of the normal vector field to a brane is chosen to be the same all way through the bulk for all the three brane. Decompose the extrinsic curvature into the traceless part and the trace part

$$e^{-\phi}K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4}g_{\mu\nu}Q \ , \quad Q = -e^{-\phi}\frac{\partial}{\partial y}\log\sqrt{-g} \ , \tag{2.12}$$

then, we obtain the basic equations;

$$\begin{aligned} e^{-\phi}\Sigma^{\nu}_{\mu,y} - Q\Sigma^{\nu}_{\mu} = -\Bigg[& {}^{(4)}R^{\nu}_{\mu} - \frac{1}{4}\delta^{\nu}_{\mu}{}^{(4)}R - D_{\mu}D^{\nu}\phi - D_{\mu}\phi D^{\nu}\phi \\ & + \frac{1}{4}\delta^{\nu}_{\mu}\left(D^{\alpha}D_{\alpha}\phi + D^{\alpha}\phi D_{\alpha}\phi\right)\Bigg] \ , \end{aligned} \tag{2.13}$$

$$\frac{3}{4}Q^2 - \Sigma^{\alpha}{}_{\beta}\Sigma^{\beta}{}_{\alpha} = \left[{}^{(4)}R\right] + \frac{12}{l^2_{A,B}} \ , \tag{2.14}$$

$$e^{-\phi}Q_{,y} - \frac{1}{4}Q^2 - \Sigma^{\alpha\beta}\Sigma_{\alpha\beta} = D^{\alpha}D_{\alpha}\phi + D^{\alpha}\phi D_{\alpha}\phi - \frac{4}{l^2_{A,B}} \ , \tag{2.15}$$

$$D_{\nu}\Sigma^{\nu}_{\mu} - \frac{3}{4}D_{\mu}Q = 0 \ . \tag{2.16}$$

And the junction conditions read

$$\left[\Sigma^{\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu}Q\right]\Bigg|_{y=0} = \frac{\kappa^2}{2}\left(-\sigma_A\delta^{\nu}_{\mu} + T^{A\nu}_{\mu}\right) \ , \tag{2.17}$$

$$\left[\Sigma^{\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu}Q\right]^{+}_{-}\Bigg|_{y=y_A} = \kappa^2\left(-\sigma_C\delta^{\nu}_{\mu} + \tilde{T}^{C\nu}_{\mu}\right) \ , \tag{2.18}$$

$$\left[\Sigma^{\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu}Q\right]\Bigg|_{y=y_B} = -\frac{\kappa^2}{2}\left(-\sigma_B\delta^{\nu}_{\mu} + \hat{T}^{B\nu}_{\mu}\right) \ . \tag{2.19}$$

The notation $[X]^{+}_{-}$ indicates that we evaluate the quantity X on both sides of the brane and take difference, $[X]^{+}_{-} = X^{+} - X_{-}$.

3. The gradient expansion method

In this section we derive the effective equations of motion using the low energy expansion method (gradient expansion method) first proposed by Kanno and Soda [13] to study our scenario. In this method, the full five dimensional equations of motion are solved, at different orders, in the bulk by performing a perturbation expansion in the metric. The parameter of expansion is defined so that the low energy regime corresponds to a regime in which the energy density (ρ) on the brane is smaller then the brane tension (σ), $\rho << \sigma$. In this regime, the parameter of expansion can be expressed as

$$\epsilon = \left(\frac{l}{L}\right)^2 \ , \tag{3.1}$$

where l is the bulk curvature scale of the AdS_5 and L is the brane curvature scale. According to the parameter of expansion (3.1), the quantities Σ_μ^ν are expanded as

$$\Sigma_\mu^{\ \nu} = \Sigma_\mu^{(0)\nu} + \Sigma_\mu^{(1)\nu} + \Sigma_\mu^{(2)\nu} + \cdots \ . \tag{3.2}$$

Then, the iteration scheme consists in writing the metric $g_{\mu\nu}$ as a sum of local tensors built out of the induced metric on the brane,

$$g_{\mu\nu}(y, x^\mu) = a^2(y, x) \left[h_{\mu\nu}(x^\mu) + g_{\mu\nu}^{(1)}(y, x^\mu) + g_{\mu\nu}^{(2)}(y, x^\mu) + \cdots \right] \ , \tag{3.3}$$

$$g_{\mu\nu}^{(n)}(y = \bar{y}, x^\mu) = 0 \ , \quad n = 1, 2, 3, ... \tag{3.4}$$

Here, $\bar{y}$ is a generic point in which the Dirichlet condition is taken on the brane.

In the following, we derive the zeroth order and first order solutions using the above scheme.

3.1 Zeroth order solutions

At zeroth order matter is neglected, we intend vacuum brane, and going at higher orders means we are considering perturbation of the vacuum solution as matter is added to the brane. The equations to solve are

$$e^{-\phi} \Sigma_{\mu,y}^{(0)\nu} - Q^{(0)} \Sigma_\mu^{(0)\nu} = 0 \tag{3.5}$$

$$\frac{3}{4} Q^{(0)2} - \Sigma_\beta^{(0)\alpha} \Sigma_\alpha^{(0)\beta} = \frac{12}{l_{A,B}^2} \ , \tag{3.6}$$

$$e^{-\phi} Q_{,y}^{(0)} - \frac{1}{4} Q^{(0)2} - \Sigma^{(0)\alpha\beta} \Sigma_{\alpha\beta}^{(0)} = -\frac{4}{l_{A,B}^2} \ , \tag{3.7}$$

$$D_\nu \Sigma_\mu^{(0)\nu} - \frac{3}{4} D_\mu Q^{(0)} = 0 \ . \tag{3.8}$$

The junction conditions are given by

$$\left[\Sigma_\mu^{(0)\nu} - \frac{3}{4} \delta_\mu^\nu Q^{(0)} \right] \Big|_{y=0} = -\frac{\kappa^2}{2} \sigma_A \delta_\mu^\nu \ , \tag{3.9}$$

$$\left[\Sigma_\mu^{(0)\nu} - \frac{3}{4} \delta_\mu^\nu Q^{(0)} \right]_-^+ \Big|_{y=y_A} = -\kappa^2 \sigma_C \delta_\mu^\nu \ . \tag{3.10}$$

$$\left[\Sigma_\mu^{(0)\nu} - \frac{3}{4} \delta_\mu^\nu Q^{(0)} \right] \Big|_{y=y_B} = \frac{\kappa^2}{2} \sigma_B \delta_\mu^\nu \ . \tag{3.11}$$

Integrating equation (3.5) and using the constraint (3.8) we obtain the solution of the traceless part of the extrinsic curvature at zeroth order

$$\Sigma_\mu^{(0)\nu} \Big|_- = \Sigma_\mu^{(0)\nu} \Big|_+ = 0 \ . \tag{3.12}$$

And the trace part of the extrinsic curvature

$$Q^{(0)} \Big|_- = \frac{4}{l_A}, \qquad Q^{(0)} \Big|_+ = \frac{4}{l_B} \ , \tag{3.13}$$

where we have inserted (3.12) into (3.6) and "$|_\pm$" denotes the solutions of each region of the bulk space-time. Inserting (3.12) and (3.13) into (2.12), the extrinsic curvature at zeroth order is

$$K^{(0)}_{\mu\nu}\Big|_\pm = \frac{e^\phi}{l_{A,B}} g^{(0)}_{\mu\nu}\Big|_\pm \ . \tag{3.14}$$

In this order, the equation (3.14) gives the evolution of the extrinsic curvature in different regions of the bulk space-time which correspond to two different values of the AdS curvature scales, l_A and l_B.

From definition of the extrinsic curvature and using equation (3.14) we need to solve two different solutions of the metric

$$-\frac{1}{2}\frac{\partial}{\partial y} g^{(0)}_{\mu\nu}\Big|_- = \frac{e^\phi}{l_A} g^{(0)}_{\mu\nu}\Big|_- \ , \qquad 0 < y < y_A \ , \tag{3.15}$$

$$-\frac{1}{2}\frac{\partial}{\partial y} g^{(0)}_{\mu\nu}\Big|_+ = \frac{e^\phi}{l_A} g^{(0)}_{\mu\nu}\Big|_+ \ , \qquad y_A < y < y_B \ . \tag{3.16}$$

Integrating equations (3.15) and (3.16) we get the zeroth order metric as

$$ds^2 = e^{2\phi(y,x)} dy^2 + a^2_\mp(y,x) h^\mp_{\mu\nu}(x) dx^\mu dx^\nu \ , \tag{3.17}$$

where

$$a_-(y,x) = e^{-d_A(y,x)/l_A} \ , \tag{3.18}$$

$$a_+(y,x) = e^{d_B(y,x)/l_B} \ . \tag{3.19}$$

Here, we have integrated from a generic point y' to y such that $d_{A,B} = \int_{y'}^{y} e^{\phi(y,x)} dy$ is the proper distance between a generic point and y. The tensor $h^\mp_{\mu\nu}$ is the induced metric tensor depending on the brane coordinates. In fact, the factor $a_\pm$ is a conformal factor that relates the metric on the branes. In this case we have

$$h^{C-}_{\mu\nu} = a^2_- h^A_{\mu\nu} \ , \tag{3.20}$$

$$h^{C+}_{\mu\nu} = a^2_+ h^B_{\mu\nu} \ . \tag{3.21}$$

where $h^{C\mp}_{\mu\nu} = g^{C-brane}_{\mu\nu}$, $h^A_{\mu\nu} = g^{A-brane}_{\mu\nu}$ and $h^B_{\mu\nu} = g^{B-brane}_{\mu\nu}$ are the induced metrics on the C-brane, A-brane and B-brane, respectively.

From the junction conditions (3.9) and (3.11) we have the fine tuning conditions for A-brane tension and B-brane tension respectively,

$$\kappa^2 \sigma_A = \frac{6}{l_A} \ , \tag{3.22}$$

$$\kappa^2 \sigma_B = -\frac{6}{l_B} \ . \tag{3.23}$$

Furthermore, the junction condition (3.10) yields

$$\kappa^2 \sigma_C = -\frac{3}{l_A}\left(\frac{\alpha - 1}{\alpha}\right) \ . \tag{3.24}$$

where we have defined $\alpha = l_B/l_A$. This equation is a fine tuning condition for C-brane tension in terms of the curvature scales l_A and l_B of the slices of AdS_5 bulk. The equation (3.22) - (3.24) also implies the relation for the brane tensions:

$$\sigma_A + 2\sigma_C + \sigma_B = 0 \; . \tag{3.25}$$

The RS I model is obtained for $\alpha = 1$ where C-brane is absent, $\sigma_C = 0$. For $\alpha < 1$ we have $\sigma_C > 0$ which correspond to inflation C-brane [16]. Various brane world models can be recovered by using relations (3.24) and (3.25) [17, 18, 19, 20].

3.2 First order solutions

The aim of this subsection is now to solve the four basic equations (2.13) - (2.16) at first order. In this order, the solution can be obtained by taking into account the terms neglected at the zeroth order. We have

$$\begin{aligned} e^{-\phi}\Sigma^{(1)\nu}_{\mu,y} - \frac{4}{l_{A,B}}\Sigma^{(1)\nu}_{\mu} = -\Big[{}^{(4)}R^{\nu}_{\mu} - \frac{1}{4}\delta^{\nu}_{\mu}{}^{(4)}R - D_\mu D^\nu \phi - D_\mu \phi D^\nu \phi \\ + \frac{1}{4}\delta^{\nu}_{\mu}\left(D^\alpha D_\alpha \phi + D^\alpha \phi D_\alpha \phi\right)\Big]^{(1)} \; , \end{aligned} \tag{3.26}$$

$$\frac{6}{l_{A,B}} Q^{(1)} = \left[{}^{(4)}R\right] \; , \tag{3.27}$$

$$e^{-\phi} Q^{(1)}_{,y} - \frac{2}{l_{A,B}} Q^{(1)} = \left[D^\alpha D_\alpha \phi + D^\alpha \phi D_\alpha \phi\right]^{(1)} \; , \tag{3.28}$$

$$D_\nu \Sigma^{(1)\nu}_{\mu} - \frac{3}{4} D_\mu Q^{(1)} = 0 \; , \tag{3.29}$$

where the superscript (1) represents the order of the gradient expansion. The junction conditions are given by

$$\left[\Sigma^{(1)\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu} Q^{(1)}\right]\Big|_{y=0} = \frac{\kappa^2}{2} T^{A\nu}_{\mu} \; , \tag{3.30}$$

$$\left[\Sigma^{(1)\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu} Q^{(1)}\right]^{+}_{-}\Big|_{y=y_A} = \kappa^2 \tilde{T}^{C\nu}_{\mu} \; . \tag{3.31}$$

$$\left[\Sigma^{(1)\nu}_{\mu} - \frac{3}{4}\delta^{\nu}_{\mu} Q^{(1)}\right]\Big|_{y=y_B} = -\frac{\kappa^2}{2} \hat{T}^{B\nu}_{\mu} \; . \tag{3.32}$$

Here, $T^{A\nu}_{\mu}$, $\tilde{T}^{C\nu}_{\mu}$ and $\hat{T}^{B\nu}_{\mu}$ are the energy-momentum tensors with the indices raised by the induced metric on the A-brane, C-brane and B-brane, respectively. We compute the Ricci tensor $\left[{}^{(4)}R^{\nu}_{\mu}\right]^{(1)}$ of a metric $a^2_{\mp} h^{\mp}_{\mu\nu}$. The Christoffel symbol is

$$\Gamma^{\rho}_{\mu\nu}(g) = \Gamma^{\rho}_{\mu\nu}(h) + D_\mu \ln a \delta^{\rho}_{\nu} + D_\nu \ln a \delta^{\rho}_{\mu} - h_{\mu\nu} D^\rho \ln a \; . \tag{3.33}$$

Here D_μ is a covariant derivative with respect to the metric $h_{\mu\nu}$. Using equation (3.33) the Ricci tensor of a metric $g_{\mu\nu} = a_-^2 h^-_{\mu\nu}$ is given by

$$\begin{aligned}\left[{}^{(4)}R^\nu_\mu(g)\right]^{(1)}_- &= \frac{1}{a_-^2}\Bigg[{}^{(4)}R^\nu_\mu(h^-) + \frac{2}{l_A}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_A + \frac{1}{l_A}\mathcal{D}_\mu d_A\mathcal{D}^\nu d_A\right) \\ &\quad + \frac{1}{l_A}\delta^\nu_\mu\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_A - \frac{2}{l_A}\mathcal{D}_\sigma d_A\mathcal{D}^\sigma d_A\right)\Bigg] . \end{aligned} \quad (3.34)$$

Contracting indices μ and ν of the equation (3.34), we obtain the expression for the Ricci scalar

$$\left[{}^{(4)}R(g)\right]^{(1)}_- = \frac{1}{a_-^2}\left[{}^{(4)}R(h^-) + \frac{6}{l_A}\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_A - \frac{1}{l_A}\mathcal{D}_\sigma d_A\mathcal{D}^\sigma d_A\right)\right] . \quad (3.35)$$

On the other hand, for the Ricci tensor and Ricci scalar of a metric $g_{\mu\nu} = a_+^2 h^+_{\mu\nu}$, respectively are given by

$$\begin{aligned}\left[{}^{(4)}R^\nu_\mu(g)\right]^{(1)}_+ &= \frac{1}{a_+^2}\Bigg[{}^{(4)}R^\nu_\mu(h^+) - \frac{2}{l_B}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_B - \frac{1}{l_B}\mathcal{D}_\mu d_B\mathcal{D}^\nu d_B\right) \\ &\quad - \frac{1}{l_B}\delta^\nu_\mu\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_B + \frac{2}{l_B}\mathcal{D}_\sigma d_B\mathcal{D}^\sigma d_B\right)\Bigg] , \end{aligned} \quad (3.36)$$

$$\left[{}^{(4)}R(g)\right]^{(1)}_+ = \frac{1}{a_+^2}\left[{}^{(4)}R(h^+) - \frac{6}{l_B}\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_B + \frac{1}{l_B}\mathcal{D}_\sigma d_B\mathcal{D}^\sigma d_B\right)\right] . \quad (3.37)$$

We also express the kinetic terms of ϕ in terms of the proper distance, the indices $-$ and $+$ have been omitted to give a general case, as follows

$$\begin{aligned}&\left[-\mathcal{D}_\mu\mathcal{D}^\nu\phi - \mathcal{D}_\mu\phi\mathcal{D}^\nu\phi + \frac{1}{4}\delta^\nu_\mu\left(\mathcal{D}^\alpha\mathcal{D}_\alpha\phi + \mathcal{D}^\alpha\phi\mathcal{D}_\alpha\phi\right)\right]^{(1)} \\ &\quad = \frac{e^{-\phi}}{a^2}\frac{\partial}{\partial y}\left[\left(\mathcal{D}_\mu\mathcal{D}^\nu d - \frac{1}{4}\delta^\nu_\mu\mathcal{D}_\sigma\mathcal{D}^\sigma d\right) - \frac{1}{l}\left(\mathcal{D}_\mu d\mathcal{D}^\nu d - \frac{1}{4}\delta^\nu_\mu\mathcal{D}_\sigma d\mathcal{D}^\sigma d\right)\right] \end{aligned} \quad (3.38)$$

In the equations above, $\mathcal{D}_\mu$ denotes the covariant derivative with respect to the induced metric $h^\mp_{\mu\nu}$ on the brane. And we have written the Ricci tensor in terms of the proper distance for a generic point on the y-axis.

Substituting (3.35) into (3.27) we obtain the solutions of negative side of the C-brane as

$$Q^{(1)}|_- = \frac{l_A}{a_-^2}\left[\frac{1}{6}{}^{(4)}R + \frac{1}{l_A}\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_A - \frac{1}{l_A}\mathcal{D}_\sigma d_A\mathcal{D}^\sigma d_A\right)\right] . \quad (3.39)$$

We are now to obtain the traceless part of the extrinsic curvature at first order $\Sigma^{(1)\nu}_\mu$. Substituting (3.35) and (3.38) into (3.26) and integrating we obtain

$$\begin{aligned}\Sigma^{(1)\nu}_\mu(y,x)|_- &= \frac{l_A}{2a_-^2}\left({}^{(4)}R^\nu_\mu - \frac{1}{4}\delta^{\nu(4)}_\mu R\right) + \frac{1}{a_A^2}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_A - \frac{1}{4}\delta^\nu_\mu\mathcal{D}_\sigma\mathcal{D}^\sigma d_A\right) \\ &\quad \frac{1}{l_A a_-^2}\left(\mathcal{D}_\mu d_A\mathcal{D}^\nu d_A - \frac{1}{4}\delta^\nu_\mu\mathcal{D}_\sigma d_A\mathcal{D}^\sigma d_A\right) + \frac{\chi^\nu_\mu|_-}{a_A^4} , \end{aligned} \quad (3.40)$$

where $\chi^{\nu}_{\mu}|_{-}$ is a integration constant which satisfy $\chi^{\mu}_{\mu}|_{-} = 0$ and $\mathcal{D}_{\nu}\chi^{\nu}_{\mu}|_{-} = 0$.

And the solutions of positive side of the C-brane are

$$Q^{(1)}|_{+} = \frac{l_B}{a_+^2}\left[\frac{1}{6}{}^{(4)}R - \frac{1}{l_B}\left(\mathcal{D}_\sigma\mathcal{D}^\sigma d_B - \frac{1}{l_B}\mathcal{D}_\sigma d_B\mathcal{D}^\sigma d_B\right)\right] , \tag{3.41}$$

$$\begin{aligned}\Sigma^{(1)\nu}_{\mu}(y,x)|_{+} = &\frac{l_B}{2a_+^2}\left({}^{(4)}R^{\nu}_{\mu} - \frac{1}{4}\delta^{\nu}_{\mu}{}^{(4)}R\right) - \frac{1}{a_B^2}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_B - \frac{1}{4}\delta^{\nu}_{\mu}\mathcal{D}_\sigma\mathcal{D}^\sigma d_B\right)\\ &\frac{1}{l_B a_+^2}\left(\mathcal{D}_\mu d_B\mathcal{D}^\nu d_B - \frac{1}{4}\delta^{\nu}_{\mu}\mathcal{D}_\sigma d_B\mathcal{D}^\sigma d_B\right) + \frac{\chi^{\nu}_{\mu}|_{+}}{a_B^4} .\end{aligned} \tag{3.42}$$

where $\chi^{\nu}_{\mu}|_{+}$ is a integration constant which satisfy $\chi^{\mu}_{\mu}|_{+} = 0$ and $\mathcal{D}_{\nu}\chi^{\nu}_{\mu}|_{+} = 0$.

The integration constants $\chi^{\nu}_{\mu}|_{-}$ and $\chi^{\nu}_{\mu}|_{+}$ are non-local terms , which corresponds to the projection on the brane of the five- dimensional Weyl tensor. Therefore they carry the information of the bulk gravitational fields. In the following subsection we give explicitly the expression of these terms.

4. The effective equations of motion on the branes

In this section, we derive the effective equations of motion on each brane, at the first order, by substituting the equations of the extrinsic curvature into the junction conditions (3.31)-(3.32). After imposing the trace of the projective Weyl tensor vanishes, $\chi^{\mu}_{\mu} = 0$, we then obtain the equations of motion for the scalar (radion) fields. We start to derive the equations of motion on the middle brane.

4.1 Solutions on the middle brane

At the middle brane (C-brane) is no Z_2 symmetry, the traceless part of the extrinsic curvature is asymmetric. Therefore it has two values in different sides of C-brane. Using equation (3.39) - (3.42), the junction condition at C-brane is written as

$$\frac{\chi^{\nu}_{\mu}|_{+}}{e^{4d_B/l_A}} - \frac{\chi^{\nu}_{\mu}|_{-}}{e^{-4d_A/l_A}} + \frac{l_A}{2}\left(\alpha - 1\right)G^{\nu}_{\mu}(h^C) = \kappa^2 T^{C\nu}_{\mu} . \tag{4.1}$$

where we have used the Einstein equations on the negative and positive sides at C-brane, $G^{\nu}_{\mu}|_{-} = G^{\nu}_{\mu}|_{+} \equiv G^{\nu}_{\mu}(h^C)$. The junction condition at A-brane is given by

$$\begin{aligned}\chi^{\nu}_{\mu}|_{-} + \frac{l_A}{2e^{2d_A/l_A}}G^{\nu}_{\mu}(h^C) - \frac{1}{e^{2d_A/l_A}}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_A - \delta^{\nu}_{\mu}\mathcal{D}_\sigma\mathcal{D}^\sigma d_A\right)&\\ +\frac{1}{l_A e^{2d_A/l_A}}\left(\mathcal{D}_\mu d_A\mathcal{D}^\nu d_A + \frac{1}{2}\delta^{\nu}_{\mu}\mathcal{D}_\sigma d_A\mathcal{D}^\sigma d_A\right) &= \frac{\kappa^2}{2e^{2d_A/l_A}}T^{A\nu}_{\mu} ,\end{aligned} \tag{4.2}$$

and the junction condition at B-brane yields

$$\begin{aligned}\chi^{\nu}_{\mu}|_{+} + \frac{\alpha l_A}{2}e^{2d_B/l_B}G^{\nu}_{\mu}(h^C) + e^{2d_B/l_B}\left(\mathcal{D}_\mu\mathcal{D}^\nu d_B - \delta^{\nu}_{\mu}\mathcal{D}_\sigma\mathcal{D}^\sigma d_B\right)&\\ +\frac{1}{l_B}e^{2d_B/l_B}\left(\mathcal{D}_\mu d_B\mathcal{D}^\nu d_B + \frac{1}{2}\delta^{\nu}_{\mu}\mathcal{D}_\sigma d_B\mathcal{D}^\sigma d_B\right) &= -\frac{\kappa^2}{2}e^{2d_B/l_B}T^{B\nu}_{\mu} ,\end{aligned} \tag{4.3}$$

where $\mathcal{D}_\mu$ is the derivative covariant with respect to the induced metric on the C-brane. To get equations (4.2) and (4.3), we have used the conformal transformation of the metric both at negative side of C-brane and A-brane and at positive side of C-brane and B-brane as follows

$$h^{C-}_{\mu\nu} = e^{-2d_A/l_A} h^{A}_{\mu\nu} \;, \tag{4.4}$$
$$h^{C+}_{\mu\nu} = e^{2d_B/l_B} h^{B}_{\mu\nu} \;. \tag{4.5}$$

Therefore, the index of $T^{A\nu}_{\mu}$ and $T^{B\nu}_{\mu}$ are the energy-momentum tensors with the indices raised by the induced metric on the both side of C-brane, while $\tilde{T}^{A\nu}_{\mu}$ and $\hat{T}^{B\nu}_{\mu}$ are he energy-momentum tensors with the indices raised by the induced metric on the A-brane and B-brane, respectively. In order to obtain the effective equations of motion at C-brane, one can subtract equation (4.2) with respect to (4.3). Then, substituting this result into equation (4.1) yields

$$\begin{aligned}
G^{\nu}_{\mu}(h^C) = {} & \frac{2\kappa^2}{l_A(\Phi_C+\alpha\Psi_C)}\left[T^{C\nu}_{\mu} + \frac{1}{2}(1+\Phi_C)T^{A\nu}_{\mu} + \frac{1}{2}(1-\Psi_C)T^{B\nu}_{\mu}\right] \\
& + \frac{1}{(\Phi_C+\alpha\Psi_C)}\Bigg[\left(\mathcal{D}_\mu\mathcal{D}^\nu\Phi_C - \delta^\nu_\mu\mathcal{D}_\sigma\mathcal{D}^\sigma\Phi_C\right) \\
& + \frac{\omega(\Phi_C)}{\Phi_C}\left(\mathcal{D}_\mu\Phi_C\mathcal{D}^\nu\Phi_C - \frac{1}{2}\delta^\nu_\mu\mathcal{D}_\sigma\Phi_C\mathcal{D}^\sigma\Phi_C\right)\Bigg] \\
& + \frac{\alpha}{(\Phi_C+\alpha\Psi_C)}\Bigg[\left(\mathcal{D}_\mu\mathcal{D}^\nu\Psi_C - \delta^\nu_\mu\mathcal{D}_\sigma\mathcal{D}^\sigma\Psi_C\right) \\
& + \frac{\omega(\Psi_C)}{\Psi_C}\left(\mathcal{D}_\mu\Psi_C\mathcal{D}^\nu\Psi_C - \frac{1}{2}\delta^\nu_\mu\mathcal{D}_\sigma\Psi_C\mathcal{D}^\sigma\Psi_C\right)\Bigg],
\end{aligned} \tag{4.6}$$

where we have defined two scalar fields

$$\Phi_C = e^{2d_A/l_A} - 1, \qquad \Psi_C = 1 - e^{-2d_B/l_B} \;, \tag{4.7}$$
$$\omega(\Phi_C) = -\frac{3}{2}\frac{\Phi_C}{(1+\Phi_C)}, \qquad \omega(\Psi_C) = \frac{3}{2}\frac{\Psi_C}{(1+\Psi_C)} \;. \tag{4.8}$$

Inserting (4.6) into (4.2) and (4.3), respectively, we obtain

$$\begin{aligned}
\chi^{\nu}_{\mu}|_{-} = {} & -\frac{\kappa^2(1+\Phi_C)}{(\Phi_C+\alpha\Psi_C)}\left[T^{C\nu}_{\mu} + \frac{1}{2}(1-\alpha\Psi_C)T^{A\nu}_{\mu} + \frac{1}{2}(1-\Psi_C)T^{B\nu}_{\mu}\right] \\
& - \frac{l_A}{2(\Phi_C+\alpha\Psi_C)}\left[(1-\alpha\Psi_C)\,P^{\nu}_{\mu}(\Phi_C) + \alpha^2\,(1+\Phi_C)\,P^{\nu}_{\mu}(\Psi_C)\right] \;,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\chi^{\nu}_{\mu}|_{+} = {} & -\frac{\alpha\kappa^2(1-\Psi_C)}{(\Phi_C+\alpha\Psi_C)}\left[T^{C\nu}_{\mu} + \frac{1}{2}(1+\Phi_C)T^{A\nu}_{\mu} + \frac{1}{2}\left(\frac{\alpha+\Phi_C}{\alpha}\right)T^{B\nu}_{\mu}\right] \\
& - \frac{\alpha l_A}{2(\Phi_C+\alpha\Psi_C)}\left[(1-\Psi_C)\,P^{\nu}_{\mu}(\Phi_C) + (\alpha+\Phi_C)\,P^{\nu}_{\mu}(\Psi_C)\right] \;,
\end{aligned} \tag{4.10}$$

where

$$
\begin{aligned}
P^{\nu}_{\mu}(\Phi_C) &= \left(\mathcal{D}_\mu \mathcal{D}^\nu \Phi_C - \delta^\nu_\mu \mathcal{D}_\sigma \mathcal{D}^\sigma \Phi_C\right) \\
&\quad + \frac{\omega(\Phi_C)}{\Phi_C}\left(\mathcal{D}_\mu \Phi_C \mathcal{D}^\nu \Phi_C - \frac{1}{2}\delta^\nu_\mu \mathcal{D}_\sigma \Phi_C \mathcal{D}^\sigma \Phi_C\right) \ , \qquad (4.11) \\
P^{\nu}_{\mu}(\Psi_C) &= \left(\mathcal{D}_\mu \mathcal{D}^\nu \Psi_C - \delta^\nu_\mu \mathcal{D}_\sigma \mathcal{D}^\sigma \Psi_C\right) \\
&\quad + \frac{\omega(\Psi_C)}{\Psi_C}\left(\mathcal{D}_\mu \Psi_C \mathcal{D}^\nu \Psi_C - \frac{1}{2}\delta^\nu_\mu \mathcal{D}_\sigma \Psi_C \mathcal{D}^\sigma \Psi_C\right) \ . \qquad (4.12)
\end{aligned}
$$

The equations (4.9) and (4.10) correspond with discontinuity of the evolution for Weyl tensor in each regions. The value $\chi^\nu_\mu|_-$ on the negative side corresponds to the evolution toward A-brane and the value $\chi^\nu_\mu|_+$ on the positive side corresponds to the evolution toward B-brane.

The effective equations of motion for both scalar fields Φ_C and Ψ_C can be obtained by using the condition $\chi^\mu_\mu|_- = 0$ and $\chi^\mu_\mu|_+ = 0$, respectively. Then we get

$$
\begin{aligned}
\Box \Phi_C &= \frac{\kappa^2}{l_A(1-\alpha)}\left[\frac{(1-\alpha)T^A + 2T^C}{2\omega(\Phi_C)+3}\right] \\
&\quad - \frac{1}{2\omega(\Phi_C)+3}\frac{d\omega(\Phi_C)}{d\Phi_C}\mathcal{D}_\sigma \Phi_C \mathcal{D}^\sigma \Phi_C \ , \qquad (4.13) \\
\Box \Psi_C &= \frac{\kappa^2}{\alpha l_A(\alpha-1)}\left[\frac{(\alpha-1)T^B + 2\alpha T^C}{2\omega(\Psi_C)+3}\right] \\
&\quad - \frac{1}{2\omega(\Psi_C)+3}\frac{d\omega(\Psi_C)}{d\Psi_C}\mathcal{D}_\sigma \Psi_C \mathcal{D}^\sigma \Psi_C \ . \qquad (4.14)
\end{aligned}
$$

We see that the effective equations of motion for the scalar fields dependent on the matter sources for two branes.

4.2 Solutions on the other branes

We now proceed to find the equations of motion on the other branes. First, we derive the equations of motion on the A-brane. Using the fact that the metric induced on the A-brane is related by $h^A_{\mu\nu} = e^{2d_A l_A} h^{C-}_{\mu\nu}$, the junction condition on the A-brane is given by

$$
\chi^\nu_\mu|_- + \frac{l_A}{2} G^\nu_\mu(h^A) = \frac{\kappa^2}{2} T^{A\nu}_\mu \ . \qquad (4.15)
$$

The junction condition at C-brane is

$$
\begin{aligned}
&\frac{\chi^\nu_\mu|_+}{e^{4d_B/l_A}} - \frac{\chi^\nu_\mu|_-}{e^{-4d_A/l_A}} + (\alpha-1)\,\frac{l_A}{2}e^{2d_A/l_A}G^\nu_\mu(h^A) \\
&+ (\alpha-1)\,e^{2d_A/l_A}\left[\left(\nabla_\mu \nabla^\nu d_A - \delta^\nu_\mu \nabla_\sigma \nabla^\sigma d_A\right) + \frac{1}{l_A}\left(\nabla_\mu d_A \nabla^\nu d_A + \frac{1}{2}\delta^\nu_\mu \nabla_\sigma d_A \nabla^\sigma d_A\right)\right] \\
&= \kappa^2 e^{2d_A/l_A} T^{C\nu}_\mu \ , \qquad (4.16)
\end{aligned}
$$

and the junction condition at B-brane yields

$$
\begin{aligned}
&\chi_\mu^\nu|_+ + \frac{\alpha l_A}{2} e^{2d_A/l_A+2d_B/l_B} G_\mu^\nu(h^A) \\
&+\alpha e^{2d_A/l_A+2d_B/l_B} \left[\left(\nabla_\mu \nabla^\nu d_A - \delta_\mu^\nu \nabla_\sigma \nabla^\sigma d_A\right) + \frac{1}{l_A} \left(\nabla_\mu d_A \nabla^\nu d_A + \frac{1}{2}\delta_\mu^\nu \nabla_\sigma d_A \nabla^\sigma d_A \right) \right] \\
&+e^{2d_A/l_A+2d_B/l_B} \left[\left(\nabla_\mu \nabla^\nu d_B - \delta_\mu^\nu \nabla_\sigma \nabla^\sigma d_B\right) + \frac{1}{\alpha l_A} \left(\nabla_\mu d_B \nabla^\nu d_B + \frac{1}{2}\delta_\mu^\nu \nabla_\sigma d_B \nabla^\sigma d_B \right) \right] \\
&+\frac{2}{l_A} e^{2d_A/l_A+2d_B/l_B} \left(\nabla_\mu d_A \nabla^\nu d_B + \frac{1}{2}\delta_\mu^\nu \nabla_\sigma d_A \nabla^\sigma d_B \right) \\
&= -\frac{\kappa^2}{2} e^{2d_A/l_A+2d_B/l_B} T_\mu^{B\nu} \ , \qquad (4.17)
\end{aligned}
$$

where ∇ is a derivative covariant with respect to the induced metric on the A-brane. Eliminating $\chi_\mu^\nu|_-$ and $\chi_\mu^\nu|_+$, the equations of motion on the A-brane is given by

$$
\begin{aligned}
G_\mu^\nu(h^A) =\ & \frac{2\kappa^2}{l_A\left(\Phi_A + \alpha(1-\Phi_A)\Psi_A\right)} \left[\frac{1}{2} T_\mu^{A\nu} + (1-\Phi_A)\left(T_\mu^{C\nu} + \frac{1}{2}(1-\Psi_A) T_\mu^{B\nu} \right) \right] \\
&+ \frac{l_A}{2\left(\Phi_A + \alpha(1-\Phi_A)\Psi_A\right)} \Big[\alpha(1-\Phi_A) P_\mu^\nu(\Psi_A) + (1-\alpha\Psi_A) P_\mu^\nu(\Phi_A) \\
&+\alpha P_\mu^\nu(\Phi_A,\Psi_A) \Big] \ , \qquad (4.18)
\end{aligned}
$$

where

$$
\begin{aligned}
P_\mu^\nu(\Phi_A) =\ & \left(\nabla_\mu \nabla^\nu \Phi_A - \delta_\mu^\nu \nabla_\sigma \nabla^\sigma \Phi_A\right) \\
&+ \frac{3}{2(1-\Phi_A)} \left(\nabla_\mu \Phi_A \nabla^\nu \Phi_A - \frac{1}{2}\delta_\mu^\nu \nabla_\sigma \Phi_A \nabla^\sigma \Phi_A \right) \ , \qquad (4.19) \\
P_\mu^\nu(\Psi_A) =\ & \left(\nabla_\mu \nabla^\nu \Psi_A - \delta_\mu^\nu \nabla_\sigma \nabla^\sigma \Psi_A\right) \\
&+ \frac{3}{2(1-\Psi_A)} \left(\nabla_\mu \Psi_A \nabla^\nu \Psi_A - \frac{1}{2}\delta_\mu^\nu \nabla_\sigma \Psi_A \nabla^\sigma \Psi_A \right) \ , \qquad (4.20) \\
P_\mu^\nu(\Phi_A,\Psi_A) =\ & \nabla_\mu \Phi_A \nabla^\nu \Psi_A + \frac{1}{2}\delta_\mu^\nu \nabla_\sigma \Phi_A \nabla^\sigma \Psi_A \ . \qquad (4.21)
\end{aligned}
$$

Here, we have defined two scalar fields as follows

$$
\Phi_A \equiv 1 - e^{-2d_A/l_A} \ , \qquad \Psi_A \equiv 1 - e^{-2d_B/l_B} \ . \qquad (4.22)
$$

Substituting (4.18) into (4.15) and (4.16), respectively, we find

$$
\begin{aligned}
\chi_\mu^\nu|_- =\ & -\frac{\kappa^2(1-\Phi_A)(1-\alpha\Psi_A)}{(\Phi_A + \alpha(1-\Phi_A)\Psi_A)} \left[\frac{1}{2} T_\mu^{A\nu} + \frac{1}{(1-\alpha\Psi_A)} \left(T_\mu^{C\nu} + \frac{1}{2}(1-\Psi_A) T_\mu^{B\nu} \right) \right] \\
&- \frac{l_A}{2(\Phi_A + \alpha(1-\Phi_A)\Psi_A)} \Big[\alpha\,(1-\Phi_A)\, P_\mu^\nu(\Psi_A) \\
&+ (1-\alpha\Psi_A)\, P_\mu^\nu(\Phi_A) + \alpha P_\mu^\nu(\Phi_A,\Psi_A) \Big] \ , \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
\chi^{\nu}_{\mu}|_{+} = & -\frac{\alpha\kappa^2}{\left(\Phi_A+\alpha(1-\Phi_A)\Psi_A\right)(1-\Psi_A)(1-\Phi_A)}\left[\frac{1}{2}T^{A\nu}_{\mu}+(1-\Phi_A)T^{C\nu}_{\mu}\right. \\
& \left.+\frac{1}{2}\frac{\alpha(1-\Phi_A)+\Phi_A}{\alpha}T^{B\nu}_{\mu}\right] \\
& -\frac{\alpha l_A}{2\left(\Phi_A+\alpha(1-\Phi_A)\Psi_A\right)(1-\Psi_A)(1-\Phi_A)}\left[\frac{1}{(1-\Phi_A)}P^{\nu}_{\mu}(\Phi_A)\right. \\
& \left.+\left(\frac{\alpha(1-\Phi_A)+\Phi_A}{(1-\Psi_A)}\right)\left(P^{\nu}_{\mu}(\Psi_A)+\frac{1}{(1-\Phi_A)}P^{\nu}_{\mu}(\Phi_A,\Psi_A)\right)\right] . \qquad (4.24)
\end{aligned}
$$

Using the conditions $\chi^{\mu}_{\mu}|_{-}=0$ and $\chi^{\mu}_{\mu}|_{+}=0$ we find the effective equations of motions for the scalar fields Φ_A and Ψ_A as follows

$$
P^{\mu}_{\mu}(\Phi_A) = -\frac{2\kappa^2(1-\Phi_A)}{l_A(1-\alpha)}\left[\frac{1}{2}(1-\alpha)T^A+T^C\right] , \qquad (4.25)
$$

$$
P^{\mu}_{\mu}(\Psi_A)+\frac{1}{(1-\Phi_A)}P^{\mu}_{\mu}(\Phi_A,\Psi_A) = \frac{2\kappa^2(1-\Psi_A)}{l_A(1-\alpha)}\left[T^C+\frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)T^B\right] \qquad (4.26)
$$

Second, we now derive the equations of motion on the B-brane. Using the relation of the metric between C-brane and B-brane the junction conditions at the A-brane is

$$
\begin{aligned}
& \chi^{\nu}_{\mu}|_{-}+\frac{l_A}{2}e^{-2d_A/l_A-2d_B/l_B}G^{\nu}_{\mu}(h^B) \\
& -\frac{l_A}{l_B}e^{-2d_A/l_A-2d_B/l_B}\left[\left(\hat{\nabla}_{\mu}\hat{\nabla}^{\nu}d_B-\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}\hat{\nabla}^{\sigma}d_B\right)-\frac{1}{l_B}\left(\hat{\nabla}_{\mu}d_B\hat{\nabla}^{\nu}d_B+\frac{1}{2}\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}d_B\hat{\nabla}^{\sigma}d_B\right)\right] \\
& +e^{-2d_A/l_A-2d_B/l_B}\left[\left(\hat{\nabla}_{\mu}\hat{\nabla}^{\nu}d_A-\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}\hat{\nabla}^{\sigma}d_A\right)+\frac{1}{l_A}\left(\hat{\nabla}_{\mu}d_A\hat{\nabla}^{\nu}d_A+\frac{1}{2}\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}d_A\hat{\nabla}^{\sigma}d_A\right)\right] \\
& +\frac{2}{l_A}e^{-2d_A/l_A-2d_B/l_B}\left(\hat{\nabla}_{\mu}d_A\hat{\nabla}^{\nu}d_B+\frac{1}{2}\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}d_A\hat{\nabla}^{\sigma}d_B\right) \\
& =\frac{\kappa^2}{2}e^{-2d_A/l_A-2d_B/l_B}T^{A\nu}_{\mu} , \qquad (4.27)
\end{aligned}
$$

and the junction condition at the C-brane is given by

$$
\begin{aligned}
& \frac{\chi^{\nu}_{\mu}|_{+}}{e^{4d_B/l_A}}-\frac{\chi^{\nu}_{\mu}|_{-}}{e^{-4d_A/l_A}}+e^{-2d_B/l_B}\left(\frac{l_B}{2}-\frac{l_A}{2}\right)G^{\nu}_{\mu}(h^B) \\
& -\frac{2e^{-2d_B/l_B}}{l_B}\left(\frac{l_B}{2}-\frac{l_A}{2}\right)\left[\left(\hat{\nabla}_{\mu}\hat{\nabla}^{\nu}d_B-\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}\hat{\nabla}^{\sigma}d_B\right)-\frac{1}{l_B}\left(\hat{\nabla}_{\mu}d_B\hat{\nabla}^{\nu}d_B+\frac{1}{2}\delta^{\nu}_{\mu}\hat{\nabla}_{\sigma}d_B\hat{\nabla}^{\sigma}d_B\right)\right] \\
& =\kappa^2e^{-2d_B/l_B}T^{C\nu}_{\mu} , \qquad (4.28)
\end{aligned}
$$

where $\hat{\nabla}$ is the covariant derivative with respect to B-brane, while the junction condition at B-brane is

$$
\chi^{\mu}_{\mu}|_{+}+\frac{\alpha l_A}{2}G^{\nu}_{\mu}(h^B) = -\frac{\kappa^2}{2}T^{B\nu}_{\mu} . \qquad (4.29)
$$

Eliminating $\chi^\nu_\mu|_-$ and $\chi^\nu_\mu|_+$ from equation (4.27)-(4.29), the equations of motion on the B-brane is given by

$$
\begin{aligned}
G^\nu_\mu(h^B) = & \frac{2\kappa^2}{l_A(\Phi_B(\Psi_B+1)+\alpha\Psi_B)}\bigg[\frac{1}{2}T^{B\nu}_\mu+(\Psi_B+1)T^{C\nu}_\mu \\
& +\frac{1}{2}(\Psi_B+1)(\Phi_B+1)T^{A\nu}_\mu\bigg] \\
& +\frac{1}{(\Phi_B(\Psi_B+1)+\alpha\Psi_B)}\bigg[(\Psi_B+1)P^\nu_\mu(\Phi_B)+(\Phi_B+\alpha)P^\nu_\mu(\Psi_B) \\
& -P^\nu_\mu(\Phi_B,\Psi_B)\bigg] \ , \qquad (4.30)
\end{aligned}
$$

where

$$
\begin{aligned}
P^\nu_\mu(\Phi_B) = & \left(\hat\nabla_\mu\hat\nabla^\nu\Phi_B-\delta^\nu_\mu\hat\nabla_\sigma\hat\nabla^\sigma\Phi_B\right) \\
& -\frac{3}{2(\Phi_B+1)}\left(\hat\nabla_\mu\Phi_B\hat\nabla^\nu\Phi_B-\frac{1}{2}\delta^\nu_\mu\hat\nabla_\sigma\Phi_B\hat\nabla^\sigma\Phi_B\right) \ , \qquad (4.31) \\
P^\nu_\mu(\Psi_B) = & \left(\hat\nabla_\mu\nabla^\nu\Psi_B-\delta^\nu_\mu\hat\nabla_\sigma\hat\nabla^\sigma\Psi_B\right) \\
& -\frac{3}{2(\Psi_B+1)}\left(\hat\nabla_\mu\Psi_B\hat\nabla^\nu\Psi_B-\frac{1}{2}\delta^\nu_\mu\hat\nabla_\sigma\Psi_B\hat\nabla^\sigma\Psi_B\right) \ , \qquad (4.32) \\
P^\nu_\mu(\Phi_B,\Psi_B) = & \hat\nabla_\mu\Phi_B\hat\nabla^\nu\Psi_B+\frac{1}{2}\delta^\nu_\mu\hat\nabla_\sigma\Phi_B\hat\nabla^\sigma\Psi_B \ . \qquad (4.33)
\end{aligned}
$$

The two scalar fields are defined as follows

$$
\Phi_B\equiv e^{2d_A/l_A}-1 \ , \qquad \Psi_B\equiv e^{2d_B/l_B}-1 \ . \qquad (4.34)
$$

The solutions for $\chi^\nu_\mu|_-$ and $\chi^\nu_\mu|_+$ can be obtained by substituting equation (4.30) into (4.27) and (4.29), respectively. Then, we get

$$
\begin{aligned}
\chi^\nu_\mu|_- = & -\frac{\kappa^2}{(\Phi_B(\Psi_B+1)+\alpha\Psi_B)(\Phi_B+1)(\Psi_B+1)}\bigg[\frac{1}{2}T^{B\nu}_\mu+(\Psi_B+1)T^{C\nu}_\mu \\
& +\frac{1}{2}(\Psi_B+1)\left(1-\frac{\alpha\Psi_B}{(\Psi_B+1)}\right)T^{A\nu}_\mu\bigg] \\
& -\frac{l_A}{2\left(\Phi_B(\Psi_B+1)+\alpha\Psi_B\right)(\Phi_B+1)^2(\Psi_B+1)^2}\bigg[\alpha(\Phi_B+1)P^\nu_\mu(\Psi_B) \\
& +\left((\Psi_B+1)-\alpha\Psi_B\right)\left((\Psi_B+1)P^\nu_\mu(\Phi_B)-P^\nu_\mu(\Phi_B,\Psi_B)\right)\bigg] \ , \qquad (4.35) \\
\chi^\nu_\mu|_+ = & -\frac{\kappa^2(\Psi_B+1)(\alpha+\Phi_B)}{(\Phi_B(\Psi_B+1)+\alpha\Psi_B)}\bigg[\frac{1}{2}T^{B\nu}_\mu+\left(\frac{\alpha}{\alpha+\Phi_B}\right)T^{C\nu}_\mu \\
& +\frac{1}{2}\left(\frac{\alpha}{\alpha+\Phi_B}\right)(\Phi_B+1)T^{A\nu}_\mu\bigg]
\end{aligned}
$$

$$-\frac{\alpha l_A}{2\left(\Phi_B(\Psi_B+1)+\alpha\Psi_B\right)}\Big[(\Psi_B+1)P^{\nu}_{\mu}(\Phi_B)+(\Phi_B+\alpha)P^{\nu}_{\mu}(\Psi_B)$$
$$-P^{\nu}_{\mu}(\Phi_B,\Psi_B)\Big] . \tag{4.36}$$

Finally, the equations of motion for the scalar fields are given by

$$P^{\mu}_{\mu}(\Phi_B)-\frac{1}{(\Psi_B+1)}P^{\mu}_{\mu}(\Phi_B,\Psi_B)=-\frac{2\kappa^2(\Phi_B+1)}{l_A}\left[\frac{1}{2}T^A+\frac{1}{(1-\alpha)}T^C\right] \tag{4.37}$$
$$P^{\mu}_{\mu}(\Psi_B)=-\frac{2\kappa^2(\Psi_B+1)}{l_A(\alpha-1)}\left[\frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)T^B+T^C\right] . \tag{4.38}$$

In the derivation of equations of motion above we first to know the dynamics on one brane. Then we know the gravity on the other branes. Since the dynamics on each branes are not independent, the transformation rules for the scalar fields are given by

$$\Phi_C=\Phi_B\ =\frac{\Phi_A}{1-\Phi_A} , \qquad \Psi_C=\Psi_A=\frac{\Psi_B}{1+\Psi_B} . \tag{4.39}$$

In the following subsection, for the realization at the first order expansion, we study the cosmological consequence of the radion dynamics. We need to derive the Friedmann equation on each branes.

4.3 The effective Friedmann equation on the branes

The metric induced on the brane is the Friedmann-Robertson-Walker (FRW) metric,

$$ds^2=-dt^2+a_i^2(t)\gamma_{mn}dx^m dx^n , \qquad i=A,B,C \tag{4.40}$$

where the time and space components of the Einstein equations are given by

$$G^0_0=-3\left[\frac{\dot{a}_i^2+k}{a_i^2}\right] , \tag{4.41}$$
$$G^m_n=-\left[\frac{2a_i\ddot{a}_i+\dot{a}_i^2+k}{a_i^2}\right]\delta^m_n , \tag{4.42}$$

and k is the spatial curvature, $k=0,\pm1$.

Assuming that the matter of the energy-momentum tensor to be $T^{i\mu}_{\nu}=-\rho^i\delta^{\mu}_{\nu}$, $(i=A,B,C)$, then the field equations on the middle brane can be written as

$$G^0_0(h^C)=-\frac{2\kappa^2\rho^C}{l_A(\Phi_C+\alpha\Psi_C)}\left[1+\frac{(1+\Phi_C)}{2\alpha_A}+\frac{(1-\Psi_C)}{2\alpha_B}\right]$$
$$+\frac{3}{(\Phi_C+\alpha\Psi_C)}\left[H(\dot{\Phi}_C+\alpha\dot{\Psi}_C)+\frac{1}{4}\left(\frac{\dot{\Phi}_C^2}{(1+\Phi_C)}-\frac{\alpha\dot{\Psi}_C^2}{(1-\Psi_C)}\right)\right] \tag{4.43}$$

$$
\begin{aligned}
G^m_m = & -\frac{2\kappa^2\rho^C}{l_A(\Phi_C+\alpha\Psi_C)}\left[1+\frac{(1+\Phi_C)}{2\alpha_A}+\frac{(1-\Psi_C)}{2\alpha_B}\right] \\
& +\frac{1}{(\Phi_C+\alpha\Psi_C)}\Bigg[2H(\dot\Phi_C+\alpha\dot\Psi_C)+(\ddot\Phi_C+\alpha\ddot\Psi_C) \\
& -\frac{3}{4}\left(\frac{\dot\Phi_C^2}{(1+\Phi_C)}-\frac{\alpha\dot\Psi_C^2}{(1-\Psi_C)}\right)\Bigg] \ ,
\end{aligned} \tag{4.44}
$$

where the equations of motion for the radion fields are

$$
\ddot\Phi_C+3H\dot\Phi_C = -\frac{8\kappa^2\rho^C(1+\Phi_C)}{3l_A}\left[\frac{1}{(\alpha-1)}-\frac{1}{2\alpha_A}\right]+\frac{\dot\Phi_C^2}{2(1+\Phi_C)} \ , \tag{4.45}
$$

$$
\ddot\Psi_C+3H\dot\Psi_C = \frac{8\kappa^2\rho^C(1-\Psi_C)}{3l_A}\left[\frac{1}{(\alpha-1)}+\frac{1}{2\alpha_B}\right]-\frac{\dot\Psi_C^2}{2(1-\Psi_C)} \ . \tag{4.46}
$$

Here, we have defined two dimensionaless parameters

$$
\alpha_A=\frac{\rho^C}{\rho^A}, \qquad \alpha_B=\frac{\rho^C}{\rho^B} \ . \tag{4.47}
$$

Using the equations (4.41) and (4.42) and eliminating $\ddot\Phi$ and $\ddot\Psi$ from (4.45) and (4.46), respectively, we get

$$
\dot H+2H^2+\frac{k}{a_C^2}=\frac{4\kappa^2}{3l_A(\alpha-1)}\rho^C \ . \tag{4.48}
$$

Integrating this equation, we obtain the Friedmann equation with dark radiation,

$$
H^2+\frac{k}{a_C^2}=\frac{2\kappa^2}{3l_A(\alpha-1)}\rho^C+\frac{C_C}{a_C^4} \ , \tag{4.49}
$$

where C_C is an integration constant.

We derive the Friedmann equation on the A-brane where the FRW metric is given by (4.40). From the equation (4.18) the time component of the Einstein equation is

$$
\begin{aligned}
G^0_0(h^A) = & -\frac{\kappa^2\rho^C}{l\theta}\left[\frac{1}{2\alpha_A}+(1-\Phi_A)\left(1+\frac{1}{2\alpha_B}(1-\Psi_A)\right)\right] \\
& +\frac{l_A}{2l\theta}\Big[\alpha(1-\Phi_A)P^0_0(\Psi_A)+(1-\alpha\Psi_A)\,P^0_0(\Phi_A)+\alpha P^0_0(\Phi_A,\Psi_A)\Big] \ ,
\end{aligned} \tag{4.50}
$$

where $l\theta$ is defined as follows

$$
l\theta \equiv \frac{l_A}{2}\left(\Phi_A+\alpha(1-\Phi_A)\Psi_A\right) \ . \tag{4.51}
$$

And the time component of the kinetic terms are given by

$$
P^0_0(\Psi_A) = 3H\dot\Psi_A-\frac{3}{4(1-\Psi_A)}\dot\Psi_A^2 \ , \tag{4.52}
$$

$$
P^0_0(\Phi_A) = 3H\dot\Phi_A-\frac{3}{4(1-\Phi_A)}\dot\Phi_A^2 \ , \tag{4.53}
$$

$$
P^0_0(\Phi_A,\Psi_A) = -\frac{3}{2}\dot\Phi_A\dot\Psi_A \ . \tag{4.54}
$$

The space component of the Einstein equation (4.18) is given by

$$G^m_m(h^A) = -\frac{\kappa^2\rho^C}{l\theta}\left[\frac{1}{2\alpha_A}+(1-\Phi_A)\left(1+\frac{1}{2\alpha_B}(1-\Psi_A)\right)\right] \\ +\frac{l_A}{2l\theta}\Big[\alpha(1-\Phi_A)P^m_m(\Psi_A)+(1-\alpha\Psi_A)\,P^m_m(\Phi_A)+\alpha P^m_m(\Phi_A,\Psi_A)\Big], \tag{4.55}$$

where

$$P^m_m(\Psi_A) = \ddot{\Psi}_A + 2H\dot{\Psi}_A + \frac{3}{4(1-\Psi_A)}\dot{\Psi}_A^2\ , \tag{4.56}$$

$$P^m_m(\Phi_A) = \ddot{\Phi}_A + 2H\dot{\Phi}_A + \frac{3}{4(1-\Phi_A)}\dot{\Phi}_A^2\ , \tag{4.57}$$

$$P^m_m(\Phi_A,\Psi_A) = -\frac{1}{2}\dot{\Phi}_A\dot{\Psi}_A\ . \tag{4.58}$$

Inserting equations (4.52)-(4.54) into (4.50) and (4.56)-(4.54) into (4.55), respectively, we obtain

$$-3\left(H^2+\frac{k}{a_A^2}\right) = -\frac{\kappa^2\rho^C}{l\theta}\left[\frac{1}{2\alpha_A}+(1-\Phi_A)\left(1+\frac{1}{2\alpha_B}(1-\Psi_A)\right)\right] \\ +\frac{l_A}{2l\theta}\Big[\alpha(1-\Phi_A)\left(3H\dot{\Psi}_A-\frac{3}{4(1-\Psi_A)}\dot{\Psi}_A^2\right) \\ +(1-\alpha\Psi_A)\left(3H\dot{\Phi}_A-\frac{3}{4(1-\Phi_A)}\dot{\Phi}_A^2\right)-\frac{3}{2}\dot{\Phi}_A\dot{\Psi}_A\Big]\ , \tag{4.59}$$

and

$$-2\left(\dot{H}-\frac{k}{a_A^2}\right)-3\left(H^2+\frac{k}{a_A^2}\right) = -\frac{\kappa^2\rho^C}{l\theta}\left[\frac{1}{2\alpha_A}+(1-\Phi_A)\left(1+\frac{1}{2\alpha_B}(1-\Psi_A)\right)\right] \\ +\frac{l_A}{2l\theta}\Big[\alpha(1-\Phi_A)\left(\ddot{\Psi}_A+2H\dot{\Psi}_A+\frac{3}{4(1-\Psi_A)}\dot{\Psi}_A^2\right) \\ +(1-\alpha\Psi_A)\left(\ddot{\Phi}_A+2H\dot{\Phi}_A+\frac{3}{4(1-\Phi_A)}\dot{\Phi}_A^2\right)-\frac{1}{2}\dot{\Phi}_A\dot{\Psi}_A\Big]\ , \tag{4.60}$$

where the equations of motion for the radion fields Φ_A and Ψ_A are obtained from the equations (4.25) and (4.26), respectively

$$\ddot{\Phi}_A = \frac{8\kappa^2\rho^C(1-\Phi_A)}{3l_A(1-\alpha)}\left[\frac{(1-\alpha)}{2\alpha_A}+1\right]-3H\dot{\Phi}_A-\frac{1}{2(1-\Phi_A)}\dot{\Phi}_A^2\ , \tag{4.61}$$

$$\ddot{\Psi}_A = -\frac{8\kappa^2\rho^C(1-\Psi_A)}{3l_A(1-\alpha)}\left[1-\frac{(1-\alpha)}{2\alpha\alpha_B}\right]-3H\dot{\Psi}_A-\frac{1}{2(1-\Psi_A)}\dot{\Psi}_A^2 \\ +\frac{1}{2(1-\Phi_A)}\dot{\Phi}_A\dot{\Psi}_A\ . \tag{4.62}$$

Substituting (4.61) and (4.62) into (4.60), respectively, we obtain

$$\dot{H}+2H^2+\frac{k}{a_A^2} = \frac{2\kappa^2\rho^A}{3l_A}\ . \tag{4.63}$$

Then, we obtain the Friedmann equation with dark radiation by integrating (4.77) on the A-brane as follows

$$H^2 + \frac{k}{a_A^2} = \frac{\kappa^2 \rho^A}{3 l_A} + \frac{C_A}{a_A^4} \ , \tag{4.64}$$

where C_A is an integration constant.

Finally, To obtain the Friedmann equation on the B-brane we use the same procedure. The Einstein equations on B-brane are given as follows

$$\begin{aligned}
-3\left(H^2 + \frac{k}{a_B^2}\right) &= -\frac{\kappa^2 \rho^C}{\widetilde{l\theta}}\left[\frac{1}{2\alpha_B} + (1+\Psi_B)\left(1 + \frac{1}{2\alpha_A}(1+\Phi_B)\right)\right] \\
&\quad + \frac{l_A}{2\widetilde{l\theta}}\Bigg[(1+\Psi_B)\left(3H\dot{\Phi}_B + \frac{3}{4(1+\Phi_B)}\dot{\Phi}_B^2\right) \\
&\quad + (\alpha + \Phi_B)\left(3H\dot{\Psi}_B + \frac{3}{4(1+\Psi_B)}\dot{\Psi}_B^2\right) + \frac{3}{2}\dot{\Phi}_A\dot{\Psi}_A\Bigg] \ ,
\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}
-2\left(\dot{H} - \frac{k}{a_B^2}\right) - 3\left(H^2 + \frac{k}{a_B^2}\right) &= -\frac{\kappa^2 \rho^C}{\widetilde{l\theta}}\left[\frac{1}{2\alpha_B} + (1+\Psi_B)\left(1 + \frac{1}{2\alpha_A}(1+\Phi_B)\right)\right] \\
&\quad + \frac{l_A}{2\widetilde{l\theta}}\Bigg[(1+\Psi_B)\left(\ddot{\Phi}_B + 2H\dot{\Phi}_B - \frac{3}{4(1+\Phi_B)}\dot{\Phi}_B^2\right) \\
&\quad + (\alpha + \Phi_B)\left(\ddot{\Psi}_B + 2H\dot{\Psi}_B - \frac{3}{4(1+\Psi_B)}\dot{\Psi}_B^2\right) + \frac{1}{2}\dot{\Phi}_A\dot{\Psi}_A\Bigg] \ ,
\end{aligned} \tag{4.66}$$

where $\widetilde{l\theta}$ is defined as follows

$$\widetilde{l\theta} \equiv \frac{l_A}{2}\left(\Phi_B(\Psi_B + 1) + \alpha\Psi_B\right) \ . \tag{4.67}$$

In order to obtain (4.65) and (4.66) we have used the components of the scalar kinetic terms:

$$P_0^0(\Psi_B) = 3H\dot{\Psi}_B + \frac{3}{4(1+\Psi_B)}\dot{\Psi}_B^2 \ , \tag{4.68}$$

$$P_0^0(\Phi_B) = 3H\dot{\Phi}_B + \frac{3}{4(1+\Phi_B)}\dot{\Phi}_B^2 \ , \tag{4.69}$$

$$P_0^0(\Phi_B, \Psi_B) = -\frac{3}{2}\dot{\Phi}_B\dot{\Psi}_B, \tag{4.70}$$

$$P_m^m(\Psi_B) = \ddot{\Psi}_B + 2H\dot{\Psi}_B - \frac{3}{4(1+\Psi_B)}\dot{\Psi}_B^2 \ , \tag{4.71}$$

$$P_m^m(\Phi_B) = \ddot{\Phi}_B + 2H\dot{\Phi}_B - \frac{3}{4(1+\Phi_B)}\dot{\Phi}_B^2 \ , \tag{4.72}$$

$$P_m^m(\Phi_B, \Psi_B) = -\frac{1}{2}\dot{\Phi}_B\dot{\Psi}_B \ . \tag{4.73}$$

The equations of motion for the scalar fields Φ_B and Ψ_B are used to eliminate the second derivative $\ddot{\Phi}_B$ and $\ddot{\Psi}_B$ in the equation (4.66), where it is given by

$$\ddot{\Psi}_B = \frac{8\kappa^2\rho^C(1+\Psi_B)}{3l_A(\alpha-1)}\left[\frac{(\alpha-1)}{2\alpha\alpha_B}+1\right] - 3H\dot{\Psi}_B + \frac{1}{2(1+\Psi_B)}\dot{\Psi}_B^2 \,, \tag{4.74}$$

$$\begin{aligned}\ddot{\Phi}_B &= \frac{8\kappa^2\rho^C(1+\Phi_B)}{3l_A}\left[\frac{1}{2\alpha_A}+\frac{1}{(1-\alpha)}\right] - 3H\dot{\Phi}_B + \frac{1}{2(1+\Phi_B)}\dot{\Phi}_B^2 \\ &\quad - \frac{1}{2(1+\Psi_B)}\dot{\Phi}_B\dot{\Psi}_B \,.\end{aligned} \tag{4.75}$$

Solving the equation (4.66) we get the Friedmann equation on the B-brane by integrating the equation below

$$\dot{H} + 2H^2 + \frac{k}{a_B^2} = -\frac{2\kappa^2\rho^B}{3l_B} \,, \tag{4.76}$$

to find

$$H^2 + \frac{k}{a_B^2} = -\frac{\kappa^2\rho^B}{3l_B} + \frac{C_B}{a_B^4} \,, \tag{4.77}$$

where C_B is an integration constant.

5. The scalar-tensor gravity

In the previous section we have derived the effective equations of motion in a three brane system. Now we show how we can write the scalar-tensor gravity using the effective equations of motion on this system. We use the solutions on the middle brane to obtain a scalar-tensor gravity with two independent scalar fields. In the following we omit subscript C of the equations that related to the middle brane. From the equation (4.6) we see that a term $\frac{l_A}{2}(\Phi+\alpha\Psi)$ can be defined as a first dimensionless scalar field,

$$l\phi \equiv \frac{l_A}{2}(\Phi+\alpha\Psi) \;, \tag{5.1}$$

where l is an arbitrary unit of length. Because the scalar fields Φ and Ψ correspond to the proper distance, the definition of the scalar field (5.1) associated with overall distance of the middle brane. Then, the second scalar field can also defined as a function of both scalar fields, we define

$$\varphi \equiv \varphi(\xi(\Phi,\Psi)) \,. \tag{5.2}$$

We intend to write the effective equations of motion on the middle brane (4.6) as follows

$$\begin{aligned}G_\mu^\nu &= \frac{\kappa^2}{l\phi}\left(\frac{1+\Phi}{2}T_\mu^{A\nu} + \frac{1-\Psi}{2}T_\mu^{B\nu} + T_\mu^{C\nu}\right) + \frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}^\nu\phi - \delta_\mu^\nu\Box\phi\right) \\ &\quad + \frac{\omega(\phi)}{\phi}\left[\left(\mathcal{D}_\mu\phi\mathcal{D}^\nu\phi - \frac{1}{2}\delta_\mu^\nu(\mathcal{D}\phi)^2\right) - \bar{\omega}(\phi)\left(\mathcal{D}_\mu\varphi\mathcal{D}^\nu\varphi - \frac{1}{2}\delta_\mu^\nu(\mathcal{D}\varphi)^2\right)\right],\end{aligned} \tag{5.3}$$

where $\omega(\phi)$ and $\overline{\omega}(\phi)$ are the arbitrary functional coupling of ϕ. The absence of mixing terms in the equation above yields the following constraints

$$\alpha\left(\frac{l_A}{2l}\right)^2-\bar{\omega}(\phi)\left(\frac{d\varphi}{d\xi}\right)^2\left(\frac{\partial\xi}{\partial\Phi}\frac{\partial\xi}{\partial\Psi}\right)=0\ . \tag{5.4}$$

Applying this constraint into equation (5.3) we have

$$\begin{aligned}G^\nu_\mu &= \frac{\kappa^2}{l\phi}\left(\frac{1+\Phi}{2}T^{A\nu}_\mu+\frac{1-\Psi}{2}T^{B\nu}_\mu+T^{C\nu}_\mu\right)+\frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}^\nu\phi-\delta^\nu_\mu\Box\phi\right)\\ &+\frac{\omega(\phi)}{\phi}\left[\left(\frac{l_A}{2l}\right)^2-\overline{\omega}(\phi)\left(\frac{d\varphi}{d\xi}\frac{\partial\xi}{\partial\Phi}\right)^2\right]\left(\mathcal{D}_\mu\Phi\mathcal{D}^\nu\Phi-\frac{1}{2}\delta^\nu_\mu(\mathcal{D}\Phi)^2\right)\\ &+\frac{\omega(\phi)}{\phi}\left[\left(\frac{\alpha l_A}{2l}\right)^2-\overline{\omega}(\phi)\left(\frac{d\varphi}{d\xi}\frac{\partial\xi}{\partial\Psi}\right)^2\right]\left(\mathcal{D}_\mu\Psi\mathcal{D}^\nu\Psi-\frac{1}{2}\delta^\nu_\mu(\mathcal{D}\Psi)^2\right)\ .\end{aligned} \tag{5.5}$$

On the other hand, by inserting (5.1) into (4.6) we obtain

$$\begin{aligned}G^\nu_\mu &= \frac{\kappa^2}{l\phi}\left(\frac{1+\Phi}{2}T^{A\nu}_\mu+\frac{1-\Psi}{2}T^{B\nu}_\mu+T^{C\nu}_\mu\right)+\frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}^\nu\phi-\delta^\nu_\mu\Box\phi\right)\\ &-\frac{3l_A}{4l\phi(1+\Phi)}\left(\mathcal{D}_\mu\Phi\mathcal{D}^\nu\Phi-\frac{1}{2}\delta^\nu_\mu(\mathcal{D}\Phi)^2\right)\\ &+\frac{3\alpha l_A}{4l\phi(1-\Psi)}\left(\mathcal{D}_\mu\Psi\mathcal{D}^\nu\Psi-\frac{1}{2}\delta^\nu_\mu(\mathcal{D}\Psi)^2\right)\ .\end{aligned} \tag{5.6}$$

The relation between the coefficients in equation (5.5) and (5.6) is

$$\frac{3l_A}{4l\phi(1+\Phi)}=\frac{\omega(\phi)}{\phi}\left(\frac{l_A}{2l}\right)^2\left[\alpha\frac{\partial\xi/\partial\Phi}{\partial\xi/\partial\Psi}-1\right]=\frac{\omega(\phi)}{\phi}\left(\frac{l_A}{2l}\right)^2\left[\alpha-\frac{\partial\xi/\partial\Psi}{\partial\xi/\partial\Phi}\right]\ . \tag{5.7}$$

and using the constraint (5.4) we get an equation of the form

$$\left[1-\frac{(1+\Phi)}{(1-\Psi)}\frac{\partial\xi/\partial\Phi}{\partial\xi/\partial\Psi}\right]\left[\alpha-\frac{\partial\xi/\partial\Psi}{\partial\xi/\partial\Phi}\right]=0\ . \tag{5.8}$$

It is easy to see that the solution of the equation (5.8) are

$$\frac{\partial\xi/\partial\Psi}{\partial\xi/\partial\Phi}=\alpha, \tag{5.9}$$

$$\frac{\partial\xi/\partial\Psi}{\partial\xi/\partial\Phi}=\frac{(1+\Phi)}{(1-\Psi)\ .} \tag{5.10}$$

The first solution yields vanishing the coefficient of (5.5) and (5.6). Then we find a solution,

$$\xi=\log\frac{(1+\Phi)}{(1-\Psi)}\ . \tag{5.11}$$

Substituting this solution into (5.4) we obtain a differential equation for φ

$$\frac{d\varphi}{d\xi} = \frac{\sqrt{\alpha}e^{\xi/2}}{(e^\xi - \alpha)}\sqrt{\frac{((1-\alpha)l_A/2 + l\phi)^2}{l^2\overline{\omega}}}\ . \tag{5.12}$$

In order for φ to be only a function of ξ, we require that in equation (5.12)

$$\left(\frac{(1-\alpha)l_A}{2} + l\phi\right)^2 = l^2\overline{\omega}(\phi)\ . \tag{5.13}$$

Then, we obtain the solution of φ,

$$e^\xi = \alpha \coth^2\left(\frac{\varphi(\xi)}{2}\right)\ . \tag{5.14}$$

Finally, the effective equations of motion on the middle brane can be written as

$$\begin{aligned} G_\mu^\nu = & \frac{\kappa^2}{l\phi}\left[\frac{3l\phi}{2l_A\omega(\phi)}\cosh^2\left(\frac{\varphi(\xi)}{2}\right)T_\mu^{A\nu} + \frac{3l\phi}{2l_B\omega(\phi)}\sinh^2\left(\frac{\varphi(\xi)}{2}\right)T_\mu^{B\nu} + T_\mu^{C\nu}\right] \\ & +\frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}^\nu\phi - \delta_\mu^\nu\Box\phi\right) + \frac{\omega(\phi)}{\phi}\left(\mathcal{D}_\mu\phi\mathcal{D}^\nu\phi - \frac{1}{2}\delta_\mu^\nu(\mathcal{D}\phi)^2\right) \\ & -\frac{9}{4\omega(\phi)}\left(\mathcal{D}_\mu\varphi\mathcal{D}^\nu\varphi - \frac{1}{2}\delta_\mu^\nu(\mathcal{D}\varphi)^2\right)\ , \end{aligned} \tag{5.15}$$

where

$$\omega(\phi) = -\frac{3}{2}\left(\frac{l\phi}{l\phi + (1-\alpha)l_A/2}\right)\ . \tag{5.16}$$

The effective action for the the middle brane corresponding to the effective equations of motion (5.1) can be rewrite as

$$\begin{aligned} S = & \frac{l}{2\kappa^2}\int d^4x\sqrt{-h}\left[\phi R - \frac{\omega(\phi)}{\phi}\mathcal{D}_\mu\phi\mathcal{D}^\mu\phi + \frac{9}{4}\frac{\phi}{\omega(\phi)}\mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi\right] \\ & + \int d^4x\sqrt{-h}\left[\mathcal{L}^C + \frac{3l\phi}{2l_A\omega(\phi)}\cosh^2\left(\frac{\varphi}{2}\right)\mathcal{L}^A + \frac{3l\phi}{2l_B\omega(\phi)}\sinh^2\left(\frac{\varphi}{2}\right)\mathcal{L}^B\right] \end{aligned} \tag{5.17}$$

where $\mathcal{L}^A$, $\mathcal{L}^C$ and $\mathcal{L}^B$ are the Lagrangian correspond to A-brane, C-brane and B-brane, respectively. This action is the scalar-tensor gravity on the brane with two scalar fields as a function of the two proper distance.

6. Conclusions

In this paper, we consider three 3-brane systems with with A and B-brane are placed at the fixed point of the orbifold whereas the C-brane is put between A- and B-brane.

We use the gradient expansion method to analyze, in the first order, the effective equations of motion, in particular the radion Lagrangian. In this case we derived the

Friedmann equation with dark energy radiation by direct elimination of the radion fields in the Einstein equations. We also derive the scalar-tensor gravity with depend on the scalar (radion) fields.

We can also generalize this scenario to the multi (more than three) 3-branes system in the low energy limit. It is also interesting, in this model, to investigate for higher order correction.

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References

[1] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, Vol. I and Vol. II, Cambridge Monographs on Mathematical Physics, Cambridge University Press, (1988)

[2] A. Lukas, B. Ovrut, K. Stelle, and D. Waldram, *Heterotic M-Theory in Five Dimension*, Nucl. Phys. B **552** (1999) 246, [arXiv: hep-th/9806051].

[3] L. Randall, and R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83** (1999) 3370, [arXiv: hep-ph/9905221];

[4] L. Randall, and R. Sundrum, *An Alternative to Compactification*, Phys. Rev. Lett. **83** (1999) 4690, [arXiv: hep-th/9906064].

[5] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, *The ekpyrotic Universe: colliding branes and the origin of the hot Big Bang*, Phys. Rev. D **64** (2001) 123522, [arXiv: hep-th/0103239].

[6] A.J. Tolley, N. Turok, and P.J. Steinhardt, *Cosmological perturbations in a big crunch/big bang space-time*, Phys. Rev. D **69** (2004) 106005, [arXiv: hepth/0306109].

[7] F.P. Zen, B.E. Gunara, Arianto, and H. Zainuddin, *On Orbifold Compactification N=2 Supergravity in Five Dimensions*, JHEP **08** (2005) 018, [arXiv:hep-th/0407112].

[8] T. Shiromizu, K. Maeda, and M. Sasaki, *The Einstein equation on the 3-Brane World*, Phys. Rev D **62** (2000) 024012, [arXiv: gr-qc/9910076].

[9] R. Maartens, *Cosmological dynamics on the brane*, Phys. Rev. D **62** (2000) 084023, [arXiv:hep-th/0004166].

[10] R. Maartens, *Brane-world gravity*, Living Rev. Rel. **7** (2004) 1, [arXiv: gr-qc/ 0312059].

[11] P. Brax and C. van de Bruck, *Cosmology and brane worlds: A review*, Class. Quant. Grav. **20** (2003) R201, [arXiv: hep-th/0303095].

[12] D. Langlois, *Brane Cosmology: An introduction*, Prog. Theor. Phys. Suppl. **148** (2003) 181 [arXiv: hep-th/0209261].

[13] S. Kanno and J. Soda, *Braneworld Effective Action at Low Energies*, Astrophys. Space Sci. **283** (2003) 639, [arXiv: gr-qc/0209086]; S. Kanno and J. Soda, *Radion and Holographic Brane Gravity*, Phys. Rev. D **66** (2002) 083506, [arXiv: hep-th/0207029].

[14] L. Cotta-Ramusino, *Low energy effective theory for brane world models*, M.Phil.Thesis, University of Portsmouth, (2004).

[15] K. Koyama and K. Koyama, *Gravitational backreaction of anti-D branes in the warped compactification*, Class. Quant. Grav. **22** (2005) 3431, [arXiv:hep-th/0505256].

[16] S. Kanno, J. Soda and D. Wand, *Braneworld Flux Inflation*, JCAP **0508** (2005) 002, [arXiv: hep-th/0506167];

[17] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross and J. Santiago, *A Three three-brane Universe:New Phenomenology for the New Millennium?*, Nucl. Phys. B **584** (2000) 313, [arXiv: hep-ph/9912552]; I. I. Kogan, S. Mouslopoulos, A. Papazoglou and G. G. Ross , *Multi-Brane Worlds and modification of gravity at large scales*, Nucl. Phys. B **595** (2001) 225, [arXiv: hep-th/0006030]; S. Mouslopoulos and A. Papazoglou, *"+-+" Brane Model Phenomenology*, JHEP 0011 (2000) 018, [arXiv: hep-ph/0003207].

[18] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, *Opening up extra dimensions at ultra-large scales* , Phys. Rev. Lett. **84** (2000) 5928, [arXiv: hep-th/0002072].

[19] J. Lykken and L. Randall, *The Shape of Gravity*, JHEP **0006** (2000) 014, [arXiv: hep-th/9908076].

[20] L. Pilo, R. Rattazzi and A. Zaffaroni, *The Fate of the Radion in Models with Metastable Graviton*, JHEP **0007** (2000) 056, [arXiv: hep-th/0004028].